Complementary discrete geometric h-field formulation for wave propagation problems

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By discretizing the magnetic field formulation for a wave propagation problem on a pair of dual interlocked grids, we obtain a discrete formulation which is complementary to the electric field formulation discretized on the same grids. In this work, we present how the h-formulation is obtained in the Discrete Geometric Approach framework; then we use it to devise an adaptive refinement scheme. Finally, considerations on the convergence of the discrete h-formulation with respect to the discrete electric field one are discussed.

***Index Terms*—Wave propagation, Complementary formulation, Adaptive mesh refinement.**

I. Introduction

ELECTROMAGNETIC WAVE PROPAGATION is usually treated by solving a problem where the unknown is the electric field $\boldsymbol{e}$. According to the Discrete Geometric Approach, this leads to associate electric voltages with primal grid edges and magnetomotive forces to the dual grid edges. However, it is well known that the electromagnetic problem can be formulated also in terms of the magnetic field $\boldsymbol{h}$. The idea behind this work is to swap the two grids. In this way we introduce a novel discrete $\boldsymbol{h}$-formulation for the wave propagation problem, where the magnetomotive forces are associated to the primal grid, while the electromotive forces to the dual one. For this formulation, both impedance boundary conditions and plane wave excitation are also introduced in a novel complementary way. A main advantage of the proposed complementary formulations is shown, by deriving an effective adaptive mesh refinement scheme.

II. Continuous wave propagation problem

From time harmonic Maxwell's equations at angular frequency ω in a bounded domain Ω

$$\nabla \times \boldsymbol{e} = -i\omega \boldsymbol{b}, \qquad \nabla \times \boldsymbol{h} = i\omega \boldsymbol{d},$$

where $\boldsymbol{d}$, $\boldsymbol{e}$, $\boldsymbol{h}$, $\boldsymbol{b}$ are respectively electric displacement, electric, magnetic and magnetic induction fields together with the constitutive relations

$$\boldsymbol{d} = \boldsymbol{\epsilon}\boldsymbol{e}, \qquad \boldsymbol{h} = \boldsymbol{\nu}\boldsymbol{b},$$

where $\boldsymbol{\nu}$ and $\boldsymbol{\epsilon}$ are symmetric positive definite material tensors, the $\boldsymbol{e}$-formulation of electromagnetic wave propagation problem

$$\nabla \times (\boldsymbol{\nu}\nabla \times \boldsymbol{e}) - \omega^2 \boldsymbol{\epsilon}\boldsymbol{e} = \mathbf{0}, \tag{1}$$

can be derived [3]. Similarly, the $\boldsymbol{h}$-formulation of the electromagnetic problem becomes

$$\nabla \times (\boldsymbol{\xi}\nabla \times \boldsymbol{h}) - \omega^2 \boldsymbol{\mu}\boldsymbol{h} = \mathbf{0}, \tag{2}$$

where $\boldsymbol{\xi} = \boldsymbol{\epsilon}^{-1}$ and $\boldsymbol{\mu} = \boldsymbol{\nu}^{-1}$.

Usual Dirichlet and Neumann boundary conditions can be applied to the problem (2) to impose *Perfect Electric Conductor* ($\boldsymbol{n} \times \boldsymbol{e} = \mathbf{0}$) and *Perfect Magnetic Conductor* ($\boldsymbol{n} \times \boldsymbol{h} = \mathbf{0}$) conditions on $\partial\Omega$ with normal $\boldsymbol{n}$. In the $\boldsymbol{h}$-formulation (2), PEC is specified as a Neumann BC, while PMC is specified as a Dirichlet BC.

Regarding the impedance boundary condition used to constrain the electric and magnetic fields on a portion of $\partial\Omega$, in [1] we showed that the discrete counterpart of the $\boldsymbol{e}$-formulation requires the construction of an *admittance* matrix $\mathbf{M}_Y$. In this work we will show that in the $\boldsymbol{h}$-formulation the construction of an *impedance* matrix $\mathbf{M}_Z$ is needed.

III. Discrete counterpart of e-formulation

Numerical treatment of (1) requires the discretization of Ω, which is obtained by means of a *primal* tetrahedral grid $\mathcal{G}$ and a barycentric dual grid $\tilde{\mathcal{G}}$ induced by $\mathcal{G}$. The electromagnetic quantities are associated with these interlocked grids as follows:

- electromotive force U_i to edges $e_i \in \mathcal{G}$;
- magnetic flux Φ_i to faces $f_i \in \mathcal{G}$;
- magnetomotive force F_i to edges $\tilde{e}_i \in \tilde{\mathcal{G}}$;
- electric flux Ψ_i to faces $\tilde{f}_i \in \tilde{\mathcal{G}}$.

Problem (1) is discretized as [4], [2]

$$(\mathbf{C}^T\mathbf{M}_\nu\mathbf{C} - \omega^2\mathbf{M}_\epsilon)\mathbf{U} = \mathbf{0}, \tag{3}$$

where $\mathbf{C}$ is the face-edge incidence matrix, $\mathbf{M}_\nu$ and $\mathbf{M}_\epsilon$ are the constitutive matrices as described in [5] and $\mathbf{U}$ is the array of the unknown voltages along the primal edges. Introducing the impedance boundary conditions, the problem

$$(\mathbf{C}^T\mathbf{M}_\nu\mathbf{C} - \omega^2\mathbf{M}_\epsilon)\mathbf{U} + i\omega\mathbf{M}_Y\mathbf{U} = -2i\omega\mathbf{F}^{b^-}, \tag{4}$$

is obtained [2], where the term $\mathbf{F}^{b^-}$ is an excitation applied on a portion of $\partial\Omega$.

IV. Discrete counterpart of $\mathbf{h}$-formulation

The idea behind the complementary wave propagation problem is to exchange the roles $\mathcal{G}$ and $\tilde{\mathcal{G}}$, by associating:

- electromotive force U_i to edges $\tilde{e}_i \in \tilde{\mathcal{G}}$;
- magnetic flux Φ_i to faces $\tilde{f}_i \in \tilde{\mathcal{G}}$;
- magnetomotive force F_i to edges $e_i \in \mathcal{G}$;
- electric flux Ψ_i to faces $f_i \in \mathcal{G}$.

In this way, complementary discrete Maxwell equations are then written as

$$\mathbf{C}\mathbf{F} = i\omega\mathbf{\Psi}, \tag{5}$$

$$\mathbf{C}^T\mathbf{U} = -i\omega\mathbf{\Phi}, \tag{6}$$

$$\mathbf{U} = \mathbf{M}_\xi\mathbf{\Psi}, \tag{7}$$

$$\mathbf{\Phi} = \mathbf{M}_\mu\mathbf{F}. \tag{8}$$

Solving (5) for $\mathbf{\Psi}$ and substituting it in (7) and then in (6), the complementary wave propagation problem results to be

$$\mathbf{C}^T\mathbf{M}_\xi\mathbf{C}\mathbf{F} - \omega^2\mathbf{M}_\mu\mathbf{F} = \mathbf{0}, \tag{9}$$

where $\mathbf{M}_\xi$ and $\mathbf{M}_\mu$ are the counterparts of $\mathbf{M}_\nu$ and $\mathbf{M}_\epsilon$, while $\mathbf{F}$ is the magnetomotive force along primal edges. Impedance boundary condition and plane wave excitation can be introduced by adding two terms to (9), obtaining

$$\mathbf{C}^T\mathbf{M}_\xi\mathbf{C}\mathbf{F} - \omega^2\mathbf{M}_\mu\mathbf{F} + i\omega\mathbf{M}_Z\mathbf{F} = 2i\omega\mathbf{U}^{b^-}, \tag{10}$$

where $\mathbf{U}^{b^-}$ is the excitation applied on a portion of $\partial\Omega$.

V. Adaptive mesh refinement

We propose an adaptive mesh refinement scheme based on the comparison of the electromagnetic energies calculated from the usual formulation and the complementary formulation respectively. The main idea behind the scheme is to refine the mesh in the subregions of Ω where the relative error between calculated energies is maximal (Fig. 1). The entire idea can be summarized in the following iterative procedure:

1) solve problems (4) and (10),
2) interpolate fields in the mesh volumes v_i, obtaining primal and complementary fields $\boldsymbol{e}_p$, $\boldsymbol{h}_p$, $\boldsymbol{e}_d$ and $\boldsymbol{h}_d$,
3) for each v_i, let $\Delta\boldsymbol{e} = \boldsymbol{e}_p - \boldsymbol{e}_d$ and $\Delta\boldsymbol{h} = \boldsymbol{h}_p - \boldsymbol{h}_d$ then calculate the energy $\Delta w = \int_{v_i} \Delta\boldsymbol{e}\cdot\epsilon\Delta\boldsymbol{e}\; dv + \int_{v_i} \Delta\boldsymbol{h}\cdot\mu\Delta\boldsymbol{h}\; dv$
4) let $\mathcal{T}$ be the set of the tetrahedra in which Ω is discretized: calculate error $\varepsilon(t) = \Delta w/w_p$ for each $v_i \in \mathcal{T}$, where $w_p = \int_{v_i} \boldsymbol{e}_p\cdot\epsilon\boldsymbol{e}_p\; dv + \int_{v_i} \boldsymbol{h}_p\cdot\mu\boldsymbol{h}_p\; dv$
5) let $k \in [0,1]$ and $\varepsilon(\mathcal{X}) = \sum_{x\in\mathcal{X}} \varepsilon(x)$,
   a) make a set $\mathcal{T}_h \subset \mathcal{T}$ such that $card\,(\mathcal{T}_h) = k\cdot card\,(\mathcal{T})$ and $\varepsilon(\mathcal{T}_h)$ is maximized,
   b) make a set $\mathcal{T}_l = \mathcal{T}\setminus\mathcal{T}_h$ that contains the remaining tetrahedra,
6) for each tetrahedron $v_i \in \mathcal{T}_h$, divide its radius by r_h,
7) for each tetrahedron $v_i \in \mathcal{T}_l$, divide its radius by r_l.

We obtained good results with $k = 0.1, r_h = 3$ and $r_l = 1.2$. Numerical experuments show that the two formulations, even if not providing lower and upper bounds of analytic quantities like in stationary problems, converge to the same solution and thus can be effectively exploited in adaptive mesh refinement.

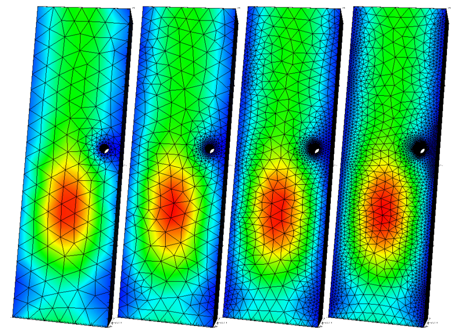

Fig. 1. Four steps of adaptive mesh refinement on a section of rectangular waveguide excited with TE_{10} mode. The adaptive scheme correctly refines the mesh near borders, where the variation of the field is higher.

VI. Numerical experiments on the convergence of the two discrete formulations

We investigated numerically, for a number of wave propagation problems, the convergence behaviour of the two formulations by calculating some energetic quantities at each refinement step. As an example, the flux of the Poynting vector across waveguide ports was evaluated by calculating

$$P_p = \frac{1}{2}Re\left[\mathbf{U}^{b^T}(\mathbf{M}_Y\mathbf{U}^b)^*\right] \qquad P_d = \frac{1}{2}Re\left[\mathbf{F}^{b*T}(\mathbf{M}_Z\mathbf{F}^b)\right]$$

for problems (4) and (10) respectively. In addition electric and magnetic energies inside Ω were also considered. From the numerical experiments, the two formulations are convergent but they fail to give an upper and a lower bound of the true solution, as happens in eddy current problems[6].

VII. Conclusions

Discrete complementary formulation of wave propagation problem was presented. Using both discrete $\boldsymbol{e}$-field and $\boldsymbol{h}$-field formulations, an adaptive refinement scheme was devised.

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